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Reduced Basis Alternatives to the Solution of Nonlinear Dynamical Systems

Chia L. Chang*
National Yunlin Institute of Technology, Touliu, Yunlin, Taiwan, Republic of China
and
John J. Engblom†
Florida Institute of Technology, Melbourne, Florida 32901

The research deals with predicting the geometric nonlinear dynamic response of a structure under impulsive loading by means of reduction methods. Techniques for reducing the number of equations to be solved, e.g., where such equations come from a finite element model, are referred to as reduction methods. The order of a dynamical system is reduced by a Rayleigh-Ritz technique using selective basis vectors. These vectors are Rayleigh-Ritz approximation functions which are not the exact eigenvectors of the system. Ritz vectors can be generated with less computational effort than needed to generate eigenvectors. Furthermore, the basis set is augmented by derivatives of Ritz vectors and updated Ritz vectors. These types of vectors are added to account for the nonlinearities of the dynamical system. The proposed reduction technique is enhanced by updating the stiffness matrix with a reasonable reassembly frequency. An error norm, which is a weighted Euclidean norm of the unbalanced force vector, is utilized to determine when updating basis vectors is necessary.

Nomenclature

C = damping matrix for the structure
$\bar{C}$ = reduced damping matrix for the structure
e = error norm defined by Eq. (44)
$\boldsymbol{I}_t$ = total unbalanced force vector at time t
K = stiffness matrix for the structure
$\bar{K}$ = reduced stiffness matrix for the structure
$K^{(d)}$ = stiffness matrix with small change in sth term of Z
K^* = effective stiffness matrix
$\bar{K}^*$ = reduced effective stiffness matrix
M = mass matrix for the structure
$\bar{M}$ = reduced mass matrix for the structure
m = number of basis vectors
N_f = frequency of updating stiffness
N_u = number of basis vectors updating
n = total number of DOF in the finite element model
$\boldsymbol{P}$ = applied loads vector
$\boldsymbol{P}^*$ = effective loads vector
$\bar{\boldsymbol{P}}^*$ = reduced effective loads vector
t_d = time duration of impulsive load
Δt = time integration step
$\boldsymbol{U}$ = displacement vector
$\dot{\boldsymbol{U}}$ = velocity vector
$\ddot{\boldsymbol{U}}$ = acceleration vector
$\boldsymbol{U}^{(d_s)}$ = displacement vector with small change in sth term of Z
$\boldsymbol{V}_c$ = basis vector related to current state
$\boldsymbol{V}_i$ = basis vector related to initial state
$\boldsymbol{Z}$ = reduced displacement vector
$\boldsymbol{Z}^{(d_s)}$ = reduced displacement vector with small change in sth term
z_s = sth components of reduced displacement vector
T_i = ith natural period of structure
Ψ = matrix of basis vectors
$\boldsymbol{\psi}_i$ = ith basis vector
ω = circular frequency in radians per second
[ ] = rectangular or square matrix
{ } = column vector

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*Associate Professor, Mechanical Engineering Department.
†Professor, Mechanical and Aerospace Engineering Department. Member AIAA.

Introduction

NONLINEAR dynamic analysis of structures has become more common because of interest in structures subjected to severe loadings and stringent design requirements. In linear analysis of structures, it has been assumed that both displacements and strains in the structure are small, i.e., geometry of the structure remains unchanged during the response. In nonlinear analysis, the above assumptions are no longer valid. The stiffness matrix of the current configuration, or tangent stiffness matrix, is a function of the current geometry (displaced position) of the structure and of material nonlinearity in stress-strain behavior. Nonlinear dynamic analysis is the prediction of the structural behavior due to time-varying loads. In this research, only geometric nonlinearity is considered.

Advances in nonlinear dynamical analysis have largely involved finding efficient formulations to modify the tangent stiffness matrix due to geometric changes (large displacements or large strains) or material nonlinearities, developing time integration schemes to accurately predict the dynamical response, improving equilibrium iteration procedures to accelerate convergence, and finally considering reduced basis techniques. To account for nonlinearities, it is necessary to update the stiffness matrix with reasonable frequency. Modification of the tangent stiffness matrix requires significant computational effort and, therefore, means that nonlinear dynamic analysis is much more demanding than linear dynamic analysis. The computer cost for obtaining the nonlinear dynamical response of a large structural system can be prohibitive. Techniques for reducing the degrees of freedom of a large dynamical system have been proposed to solve reduced order nonlinear dynamical problems. The objective of reduction techniques is to solve for the response of the nonlinear dynamical system using a reduced set of equations with sufficient accuracy.

The nonlinear dynamical response of a structure has been traditionally obtained by the direct time integration method. Equations of motion are written in incremental form. The variation of displacements, velocities, and accelerations at each node of the model are assumed at each step. Then the effective stiffness, which includes the tangent stiffness matrix and the effect of mass matrix and damping matrix, is formed. Also, the effective loads at each step, which include externally applied loads and the effect of acceleration-dependent inertia forces and velocity-dependent damping forces, are estimated. The governing equations of motion then can be solved recursively as in static analysis. Errors from the linearization at each step are expected. Either selecting a small enough time step or some iteration technique for system equilibrium at each time step is necessary to ensure an accurate prediction.

A reduced basis technique for solving nonlinear dynamic equations is the modal superposition method. Mathematically, the method involves a coordinate transformation from the finite element displacements to the reduced (or generalized) displacements. Traditionally, eigenvectors of a dynamical system are used as basis vectors (or modes) of the complete system. In this approach, a set of n-coupled equations of motion is transformed into a set of $m(m << n)$ uncoupled equations. Each uncoupled equation can be solved by any suitable time integration method. The nodal displacements and stresses are then given as the superposition of the normal modes. The modal superposition method used to obtain nonlinear dynamical response requires extensive computational effort to modify the eigenvectors associated with the changing configuration of the model. Solving the eigenproblem, even for a subset of eigenvectors, requires significant computational operations. This fact is especially important for nonlinear problems where the basis vectors must be modified in some way as the solution proceeds. This approach minimizes the computational operations in decomposition and back-substitution of the effective stiffness matrix because the equations are solved in much smaller order. If a few modes need to be considered during the response and the selected modes can be updated by an efficient algorithm, the method can compete with the direct time integration method.

This research deals with predicting the nonlinear dynamical response of a structure under impulsive loading by means of an alternative reduced basis technique. The order of the structural system is reduced by a Rayleigh-Ritz technique using selective basis vectors. The physical displacement of the full system is obtained by re-expansion of the solution from the reduced system. The approach is similar to the modal superposition method, except the basis vectors are not limited to eigenvectors of the system. Ritz vectors, which are not the exact eigenmodes of a given dynamical system, proposed by Wilson et al.,[1] are chosen as basis vectors. It has not been proven that the modal superposition method using exact eigenmodes produces better results than those with other well conceived sets of orthogonal vectors. Ritz vectors have two primary advantages over eigenvectors: 1) they are generated with significantly less computational effort than that required to obtain eigenvectors. The reduction of computer cost is greater for a large structural system; and 2) they can account for spatial load distribution, which the generation of eigenvectors ignores. To better account for nonlinearities of the structural system, derivatives of basis vectors with respect to the generalized coordinates are added to the basis set. Only first-order derivatives are included in the basis; however, higher-order derivatives have been considered. The motivation of this research is, therefore, to develop an efficient reduced basis procedure for obtaining the response of dynamical systems to impulsive loads.

Wilson et al.[1] suggest that Ritz vectors are an efficient means of transforming a dynamical system into its reduced (or generalized) coordinates. They considered the transient response of linear systems; in some cases they obtained more accurate results with Ritz vectors than with exact eigenvectors as the basis vector set. However, the generation of Ritz vectors avoids the lengthy computation needed to generate exact eigenvectors. They solved for the displacement response of typical steel frame structures with Ritz vectors. Comparable numerical results were obtained with a subspace of Ritz vectors rather than a subspace of exact eigenvectors.

Noor and Peters[2] used a nonlinear static solution and the derivatives of the static response with respect to a path parameter (a loading or displacement parameter) as basis vectors to predict the nonlinear static response of structures. A Euclidean norm of the residual vector was the error measurement of the system. If the error norm is less than a preset tolerance, the solution is continued; otherwise, an updated set of basis vectors is generated. Reference 2 applied reduction techniques only to predicting the nonlinear static response of structures.

McNamara and Marcal[3] applied the corrected incremental equations of motion; i.e., the corrected (or unbalanced) forces related to the previous step were added to the incremental applied force vector, to obtain nonlinear dynamical response solutions. They concluded that the corrected incremental scheme gave more stable and accurate solutions than the simple incremental equations of motion. In order to obtain accurate solutions, they noted that the correction terms should be computed more frequently than the reassembly of the equations.

Mondkar and Powell[4] investigated the static and dynamic response of nonlinear structures by both step-by-step and iteration schemes. The nonlinear solutions of cantilever beam, clamped beam, and shallow arch modeled by isoparametric elements were obtained by various schemes. References 3 and 4 utilized various direct-time integration schemes to solve for the nonlinear dynamical response of structures.

Horri and Kawahara[5] used the modal superposition method to solve nonlinear dynamical problems for simple structures subjected to harmonic excitation. Eigenvalues and the associated eigenvectors were updated at every time increment.

Nickell[6] employed a local modal superposition principle to solve nonlinear dynamical problems. The principle states that small harmonic motion may be superimposed on large static motion and that small forced motion may be represented in terms of the nonlinear (tangent stiffness) frequency spectrum. The subspace iteration scheme was employed to update eigenvectors at each time step. Geometrically nonlinear problems were solved by the principle of local modal superposition.

Morris[7] chose eigenvectors as basis vectors and applied the modal superposition method to the calculation of the nonlinear dynamical response of structures. Three-dimensional cable structures, an unstiffened suspension bridge, and a three-dimensional elastic-plastic frame were chosen to exemplify both the advantages and disadvantages of the modal superposition method.

Noor[8] applied the lowest vibration modes of the initial state of the structure and lowest vibration modes of the nonlinear steady state of the structure as basis vectors to predict the nonlinear dynamical response of step loading on a clamped, shallow, spherical cap subjected to a point load at the apex. It was shown that reasonably accurate solutions can be obtained using this reduction method.

Idelsohn et al.[9] augmented Ritz vectors by adding derivatives of these vectors with respect to generalized displacements to treat nonlinear dynamical problems. An error measure that indicates the need of performing a basis updating was proposed. References 5–9 were limited to treating nonlinear dynamical response of either step or harmonic loads by various reduction techniques.

Chang and Engblom[10] utilized a Ritz vector reduction technique to predict structural response produced by impulsive loads. Only lower modes of excitation were considered.

Dynamic Response: Reduced Basis Approach

Nonlinear Equations of Motion

The equations of motion governing the nonlinear dynamical system can be written in incremental form as

$$M\Delta\ddot{U}_{t+\Delta t} + C_t\Delta\dot{U}_{t+\Delta t} + K_t\Delta U_{t+\Delta t} = \Delta P_{t+\Delta t} + I_t \quad (1)$$

and

$$I_T = P_T - M\ddot{U}_t - C_t\ddot{U}_t - F_t$$

where M = the mass matrix, C_t the damping matrix at time t, K_t = the nonlinear stiffness matrix at time t, $\Delta P_{t+\Delta t}$ = the incremental applied loads vector at time $t + \Delta t$, I_t = the total unbalanced force vector at time t, F_t = vector of nodal point forces corresponding to the internal element stress at time t, and $\Delta\ddot{U}_{t+\Delta t}$, $\Delta\dot{U}_{t+\Delta t}$ and $\Delta U_{t+\Delta t}$ the unknown incremental acceleration, velocity and displacement vectors at time $t + \Delta t$. The constant-average-acceleration method (also called the trapezoidal rule) of the Newmark integration scheme is considered here because of its relatively good stability and accuracy characteristics. The basic assumptions of the trapezoidal rule are

$$\begin{aligned} \dot{U}_{t+\Delta t} &= \dot{U}_t + \frac{\Delta t}{2}(\ddot{U}_{t+\Delta t} + \ddot{U}_t) \\ U_{t+\Delta t} &= U_T + \frac{\Delta t}{2}(\dot{U}_{t+\Delta t} + \dot{U}_t) \end{aligned} \tag{2}$$

Using Eq. (2) and the relation $\Delta U_{t+\Delta t} = U_{t+\Delta t} - U_t$, Eq. (1) gives

$$K^*_{t+\Delta t}\Delta U_{t+\Delta t} = P^*_{t+\Delta t} \tag{3}$$

Here, the effective stiffness matrix $K^*_{t+\Delta t}$ is given by

$$K^*_{t+\Delta t} = K_t + \frac{2C_t}{\Delta t} + \frac{4M}{\Delta t^2} \tag{4}$$

and the effective load vector $P^*_{t+\Delta t}$ is defined as

$$P^*_{t+\Delta t} = \Delta P_{t+\Delta t} + I_t + M\left(\frac{4\dot{U}_t}{\Delta t} + 2\ddot{U}_t\right) + 2C_t\dot{U}_t \tag{5}$$

The displacement increment ΔU can be approximated by m linearly independent basis vectors, which leads to the coordinate transformation

$$\Delta U \approx \Psi Z(t) \approx \sum_{j=1}^{m} \psi_j z_j \tag{6}$$

where ψ is a matrix composed of $m(m << n)$ basis vectors and $Z(t)$ represents a set of reduced coordinates. Note that ΔU and $\Delta U_{t+\Delta t}$ are used interchangeably herein to represent the incremental displacement vector. Based on a Rayleigh-Ritz technique, we can obtain the governing equations of motion in a reduced coordinate space as

$$\bar{K}^*_{t+\Delta t}Z = \bar{P}^*_{t+\Delta t} \tag{7}$$

and, the reduced effective matrix $\bar{K}^*_{t+\Delta t}$ is defined as

$$\bar{K}^*_{t+\Delta t} = \Psi^T K^*_{t+\Delta t}\Psi \tag{8}$$

or

$$\bar{K}^*_{t+\Delta t} = \frac{4}{\Delta t^2}\bar{M} + \frac{2}{\Delta t}\bar{C}_t + \bar{K}_t \tag{9}$$

Also terms are given as

$$\begin{aligned} \bar{K}_t &= \Psi^T K_t \Psi \\ \bar{C}_t &= \Psi^T C_t \Psi \\ \bar{M} &= \Psi^T M \Psi \end{aligned} \tag{10}$$

The reduced effective load vector $\bar{P}^*_{t+\Delta t}$ is written as

$$\bar{P}^*_{t+\Delta t} = \Psi^T P^*_{t+\Delta t} \tag{11}$$

Equation (7) can be solved by virtually any suitable numerical method; remember that this is a very small equation set because the efficiency comes from using a small number of basis vectors. The approximate incremental displacement response for the complete model is then obtained by the transformation in Eq. (6).

Then the total displacement vector, total velocity vector, and total acceleration vector at time $t + \Delta t$ equal

$$\begin{aligned} \ddot{U}_{t+\Delta t} &= -\ddot{U}_t + \frac{4}{\Delta t}\dot{U}_t + \frac{4}{(\Delta t)^2}\Delta U \\ \dot{U}_{t+\Delta t} &= \frac{\Delta t}{2}\ddot{U}_{t+\Delta t} + \frac{\Delta t}{2}\ddot{U}_t + \dot{U}_t \\ U_{t+\Delta t} &= U_t + \Delta U \end{aligned} \tag{12}$$

Assume that the incremental displacement vector ΔU can be expanded by the Taylor series as

$$\Delta U = \frac{\partial \Delta U}{\partial z_i} z_i + \frac{\partial^2 \Delta U}{\partial z_i \partial z_j}\frac{z_i z_j}{2} + \frac{\partial^3 \Delta U}{\partial z_i \partial z_j \partial z_k}\frac{z_i z_j z_k}{6} + \ldots \tag{13}$$

The derivatives of the incremental displacement vector with respect to the generalized coordinates can be estimated from Eq. (6)

$$\frac{\partial \Delta U}{\partial z_i} = \psi_i + \frac{\partial \psi_j}{\partial z_i} z_i \tag{14}$$

The second-order derivatives of the incremental displacement vector can be estimated by differentiating Eq. (14)

$$\frac{\partial^2 \Delta U}{\partial z_i \partial z_j} = \frac{\partial \psi_i}{\partial z_j} + \frac{\partial \psi_j}{\partial z_i} + \frac{\partial^2 \psi_j}{\partial z_i \partial z_j} z_j \tag{15}$$

And the third-order derivatives of the incremental displacement vector can be calculated by further differentiating Eq. (15)

$$\frac{\partial^3 \Delta U}{\partial z_i \partial z_j \partial_k} = \frac{\partial^2 \psi_i}{\partial z_j \partial z_k} + \frac{\partial^2 \psi j}{\partial z_i \partial z_k} + \frac{\partial^2 \psi_k}{\partial z_i \partial z_j} + \frac{\partial^3 \psi_j}{\partial z_i \partial z_j \partial z_k} z_j \tag{16}$$

These derivatives are evaluated at time $= t$ or $z = 0$, which gives

$$\frac{\partial \Delta u}{\partial z_i} = \psi_i \tag{17}$$

$$\frac{\partial^2 \Delta U}{\partial z_i \partial z_j} = \left(\frac{\partial \psi_i}{\partial z_j} + \frac{\partial \psi_j}{\partial z_i}\right) \tag{18}$$

$$\frac{\partial^3 \Delta U}{\partial z_i \partial z_j \partial z_k} = \left(\frac{\partial^2 \psi_i}{\partial z_j \partial z_k} + \frac{\partial^2 \psi_k}{\partial z_i \partial z_k} + \frac{\partial^2 \psi_k}{\partial z_i \partial z_j}\right) \tag{19}$$

Then, the incremental displacement vector can be written as the linear combination of the basis vectors and their derivatives evaluated at time $= t$ as

$$\Delta U \approx \left[\psi_i \left(\frac{\partial \psi_i}{\partial z_j} + \frac{\partial \psi_j}{\partial z_i} \right) \left(\frac{\partial^2 \psi_i}{\partial z_j \partial z_k} + \frac{\partial^2 \psi_j}{\partial z_i \partial z_k} + \frac{\partial^2 \psi_k}{\partial z_i \partial z_j} \right) \right] \begin{pmatrix} z_i \\ \frac{z_i z_j}{2} \\ \frac{z_i z_j z_k}{6} \end{pmatrix} \tag{20}$$

Generation of Ritz Vectors

For the initial state ($t = 0$), the basis vectors are generated as

$$K_t \psi_r^* = P, \qquad r = 1 \tag{21}$$

and

$$K_t \psi_r^* = M \psi_{r-1}, \qquad r > 1 \tag{22}$$

The vectors are orthonormalized by the following procedure:

$$\psi_r' = \psi_r^* - \sum_{i=1}^{r-1} C_i \psi_i \tag{23}$$

where

$$c_i = \psi_i^T M \psi_r^* \tag{24}$$

$$\psi_r = \frac{\psi_r'}{\psi_r'^T M \psi_r'} \tag{25}$$

The basis vectors can also be augmented by selected derivatives of basis vectors with respect to the generalized coordinates and updated Ritz vectors generated from the current configuration of the dynamic system.

If the error norm estimate is greater than a prescribed tolerance, updated Ritz vectors can be added to the basis vectors to improve the response. During the dynamic response, the approximation method is applied at each step to estimate the incremental displacements. Some differences between the reduced system and the full system can gradually accumulate. In order to avoid any progressive deterioration of basis vectors, therefore, the initially computed vectors are kept through the response.

For the forced-vibration era ($t \leq t_d$), here t_d represents time duration of the impulsive load, the generation of updated Ritz vectors is identical to generation of the initial Ritz vectors. When generating Ritz vectors, the first Ritz vector is a function of spatial loading. Since the system is loaded during the forced-vibration era, Ritz vectors are updated during the loading on the basis of the load vector.

In the residual vibration era ($t > t_d$), the first basis vector of the current state can be generated from Eq. (21), assuming that the same effect of loading condition as during the forced-vibration era are supplied to the system. Alternatively, the first Ritz vector can be generated from a modification of Eq. (21) as

$$K_t \psi_1^* = M \ddot{U}_{t-1} \tag{26}$$

where $\ddot{U}_{t-1}$ is the acceleration vector at time $t - 1$. The force vector $M\ddot{U}_{t-1}$ used to generate the first updated Ritz vector represents dynamic forces at the previous step. The new basis set obtained by adding updated Ritz vectors should reflect not only the spatial loading distribution of the system, but also the approximate eigenspectrum of the full system. The generation of additional basis vectors proceeds in the same manner as used in generating the initial state, i.e., Eqs. (22–25).

Estimation of Derivatives

Based on generation of Ritz vectors and neglecting procedures of orthogonalization and normalization

$$K_t \psi_i = P, \qquad i = 1 \tag{27}$$

$$K_t \psi_i = M\psi_{i-i}, \qquad i > 1 \tag{28}$$

Take derivatives of the above equation with respect to the generalized coordinates z_i

$$\frac{\partial K_t}{\partial z_j} \psi_i + K_t \frac{\partial \Psi_i}{\partial z_j} = 0, \qquad i = 1 \tag{29}$$

$$\frac{\partial K_t}{\partial z_j} \psi_i + K_t \frac{\partial \psi_i}{\partial z_j} = M \frac{\partial \psi_{i-1}}{\partial z_j}, \qquad i > 1 \tag{30}$$

Then, the first-order derivatives of Ritz vectors can be written as

$$\frac{\partial \psi_i}{\partial z_j} = K_t^{-1} \left(-\frac{\partial K_t}{\partial z_j} \psi_i \right), \qquad i = 1 \tag{31}$$

$$\frac{\partial \psi_i}{\partial z_j} = K_t^{-1} \left(-\frac{\partial K_t}{\partial z_j} \psi_i + M \frac{\partial \psi_{i-1}}{\partial z_j} \right), \qquad i > 1 \tag{32}$$

The first-order derivatives of Ritz vectors can be estimated from the above formulation. Alternatively, the first-order derivatives can be simplified by neglecting the inertial terms. Then the derivatives can be computed as

$$\frac{\partial \psi_i}{\partial z_j} = K_t^{-1} \left(-\frac{\partial K_t}{\partial z_j} \psi_i \right) \tag{33}$$

Applying the chain rule to differentiate Eq. (32), the second-order derivatives of Ritz vectors can be written as

$$\frac{\partial^2 \psi_i}{\partial z_j \partial z_k} = K_t^{-1} \left(-\frac{\partial^2 K_t}{\partial z_j \partial z_k} \psi_i - \frac{\partial K_t}{\partial z_j} \frac{\partial \psi_i}{\partial z_k} - \frac{\partial K_t}{\partial z_k} \frac{\partial \psi_i}{\partial z_j} \right) \tag{34}$$

This form for second-order derivatives ignores the inertial terms. Some of the possible candidates for basis vectors would include eigenvectors, Ritz vectors, and derivatives of the above vectors with respect to the generalized coordinates.

Numerical evaluation of the derivatives of the basis vectors requires definition of the change in the reduced displacement vector due to a change in the sth term

$$Z^{(d_s)} = (0, 0 \ldots, \Delta z_s, \ldots, 0) \tag{35}$$

where the components of $Z^{(d_s)}$ equal zero, except in the sth term. Δz_s should be chosen small enough to accurately estimate the derivatives

$$\Delta U^{(d_s)} = \Psi Z^{(d_s)} \tag{36}$$

where $\Delta U^{(d_s)}$ is the incremental displacement vector due to a small change in sth term of the reduced displacement vector. This gives a total displacement vector reflecting the small change in the sth term as

$$U^{(d_s)} = U_t + \Delta U^{(d_s)} \tag{37}$$

Now the stiffness matrix, based on total displacement vector $U^{(d_s)}$, is obtained through a finite element program. We have

$$\Delta K = K^{(d_s)} - K_t \tag{38}$$

here, ΔK is the change in the "global" stiffness due to a change in the reduced displacement term Δz_s. Now define the approximate derivative of the basis vector with the use of a finite difference representation. Then the simplified formulation of the first-order derivatives can be written as

$$\frac{\partial \psi_r}{\partial z_s} \approx -K_t^{-1} \frac{\Delta K}{\Delta z_s} \psi_r \tag{39}$$

K_t^{-1} is obtained by an LDL^T decomposition of the current stiffness matrix. The first order derivatives with the inertial terms included can also be written as

$$\frac{\partial \psi_r}{\partial z_s} \approx K_t^{-1} \left[\frac{\Delta K}{\Delta z_s} \psi_r + M \frac{\partial \psi_{r-1}}{\partial z_s} \right] \tag{40}$$

The second-order derivatives can be written based on Eq. (34) as

$$\frac{\partial^2 \psi_i}{\partial z_j \partial z_k} \approx K_t^{-1} \left(-\frac{\Delta^2 K}{\Delta z_j \Delta Z_k} \psi_i - \frac{\Delta K}{\Delta z_j} \frac{\partial \psi_i}{\partial z_k} - \frac{\Delta K}{\Delta z_k} \frac{\partial \psi_i}{\partial z_j} \right) \tag{41}$$

Here, $\Delta^2 K / \Delta z_j \Delta z_k$ is estimated by a finite difference approach as

$$\frac{\Delta^2 K}{\Delta z_k \Delta z_k} \approx \frac{1}{\Delta z_k} \left[\frac{K(U_t + \Delta U^{(ds)} + \Delta U^{(dk)}) - K(U_t + \Delta U^{(dk)})}{\Delta z_j} - \frac{K(U_t + \Delta U^{(dj)} - K(U_t)}{\Delta z_j} \right] \tag{42}$$

Damping Matrix

Generally a damping matrix will be computed in terms of damping ratios established from experiments. The equivalent viscous damping force, $C_t \dot{U}$, account for the total energy dissipation of the structure in motion. In the present formulation viscous damping is introduced as Rayleigh damping or proportional damping. The damping matrix is then expressed by

$$C_t = \alpha M + \beta K_t \tag{43}$$

where α and β are constants which can be based on the damping ratios at two unequal natural frequencies of vibration. The lower modes are primarily damped by the mass proportional damping and the higher modes by the stiffness proportional damping.

Updating Basis

A weighted error norm of the unbalance force vector[9] that shows the need for updating basis vectors is used as an error tolerance in the following examples. The error norm is represented as

$$e = \frac{1}{n} \frac{\|I_{t+\Delta t}\|}{(\|P_{t+\Delta t}\| + \|M\ddot{U}_{t+\Delta t}\|)} \tag{44}$$

where n represents the number degrees of freedom of the full system; $\|I_{t+\Delta t}\|$ is the norm of the unbalance force vector, $\|P_{t+\Delta t}\|$ is the norm of the total applied load vector at time $t + \Delta t$; and $\|M\ddot{U}_{t+\Delta t}\|$ is the norm of the inertial force at time $t + \Delta t$. When the basis vectors do not well represent the displacement vector, the misrepresentation leads to errors in the unbalance force vector. This is because the unbalance forces are calculated using the local (element) stiffness, which depend upon the current total displacement state. The denominator in Eq. (44) serves to normalize the error norm, i.e., to compare the unbalance to some definition of total loading. Note that including the inertial term in Eq. (44) gives the error norm utility in the residual response regime.

If the error norm e is less than a prescribed tolerance, the basis vectors are not updated; otherwise some new basis vectors related to the current state can be added to the basis vectors. When updating the basis vectors, only the basis vectors related to the current state are updated; the basis vectors related to the initial state remain unchanged.

Condition Number

When applying a reduction technique, the eigen-spectrum of the reduced equations is changed to a different spectrum than exhibited by the full system. A quantitative measure of possible solution divergence of the reduced system is provided by the condition number of the reduced system. The condition number is defined by

$$\text{cond} = \frac{\omega_{\max}^2}{\omega_{\min}^2} \tag{45}$$

in which $\omega_{\max}^2$ and $\omega_{\min}^2$ are the maximum and minimum eigenvalues of the reduced system. Reference 11 shows that the solution accuracy for a t-digit precision computer, assuming s-digit precision in the solution, can be written as

$$s \geq t - \log_{10}(\text{cond}) \tag{46}$$

We may expect that the solution becomes divergent when the minimum eigenvalue is nearly zero, i.e., the system can almost undergo rigid body motion, or a negative condition number occurs, i.e., the system becomes negative definite. A large condition number also indicates a divergent solution. In the nonlinear formulation, errors accumulate during the time integration. The drift caused by the accumulated error, therefore, can significantly affect solution accuracy.

Computational Procedure

The basic computational procedure of the reduction method by Ritz vectors follows: 1) generate basis vectors; 2) form reduced equations of motion; 3) compute error norm; and 4) update basis vectors, if necessary.

Numerical Examples

The examples considered focus on studying the response of beams subjected to impulsive loads. Various combinations of Ritz vectors and their derivatives provide insight into the utility of the formulation. It is noted that the effects of damping are not considered in these examples, i.e., α and β in Eq. (43) are set equal to zero. Furthermore, the effect of including inertial terms in the basis derivative vector calculation, see Eq. (32) vs. Eq. (33), has been considered in the response calculations. Even with multiple modes excitation, the effect of including inertial terms to evaluate derivative vectors is insignificant. Therefore, the simplified formulation without inertial terms is used to obtain the results presented here.

Clamped Beam

Results for the dynamical response of a clamped beam subjected to an impulsive load are analyzed. Note that the load is applied at the midspan of the beam. The finite element model of the beam consists of five eight-node plane stress elements along the half span, with 3 × 3 Gauss quadrature integration. A refined model consisting of 10 elements along the half span is also considered. The clamped beam under step loading was analyzed by Mondkar et al.[4] by using various direct integration techniques for an unreduced set of equations. The material was assumed to be isotropic and linear elastic. The values of the parameters used for the beam are taken to be: total length = 20 in.; cross-sectional area = $\frac{1}{8}$ in. × 1 in. (thickness = $\frac{1}{8}$ in. and width = 1 in.); Young's modulus = 30 × 10^6 psi; Poisson's ratio = 0.0 and density = 0.098 lb/in.^{3}. This half-span model consists of 40 DOF. The nodes at the fixed end are all constrained in the horizontal direction. While upper and lower nodes at the fixed end can move in the vertical direction. Due to symmetry, all nodes in the center of the beam are constrained from moving in the horizontal direction.

Case 1: Load Duration Near Fourth Natural Period

The finite element model utilized in this example is the five-element half-span model. The specific parameters used for the response are as follows: centrally applied load, and 2400 lbs; and time duration of load, 0.48 ms, which is about 75% of the fourth natural period of the undeformed beam and time integration step = 4 μs. The stiffness matrix is updated at every tenth time increment in this case.

Results given in Fig. 1 demonstrate the effectiveness of including derivatives in the basis set. Three basis sets are considered and comparison made to the full system nonlinear response as well as to linear response. The first basis set uses four initial Ritz vectors, the second basis set uses four initial Ritz vectors and one derivative, while the third set uses four initial Ritz vectors and six derivatives. The initial Ritz vectors are simply the first four calculated, see Eqs. (21–25), i.e., they are not selected in any special manner or out of sequence. When one derivative is included in the basis set, it is simply the derivative of the first Ritz vector with respect to the first generalized coordinate. Additional derivatives are of the first three Ritz vectors with respect to the first two generalized coordinates. Results clearly demonstrate the benefit of including more derivatives in the basis set.

The effect of simply updating the basis vectors instead of using derivatives is given in Fig. 2. The basis set is augmented by current Ritz vectors instead of using derivatives. These results indicate that simply adding current Ritz vectors to the

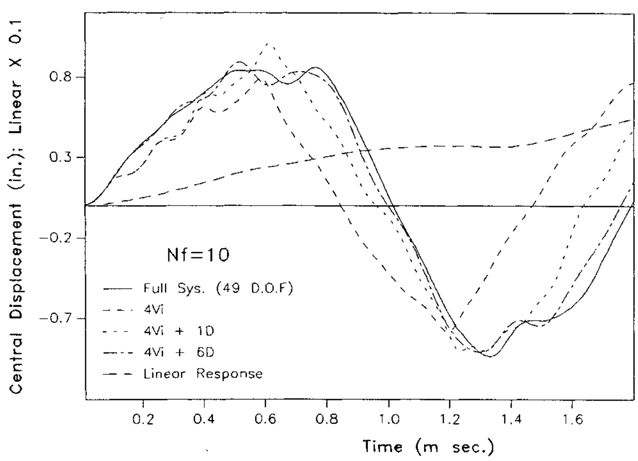

Fig. 1 Nonlinear dynamic response, effect of adding derivatives, case 1.

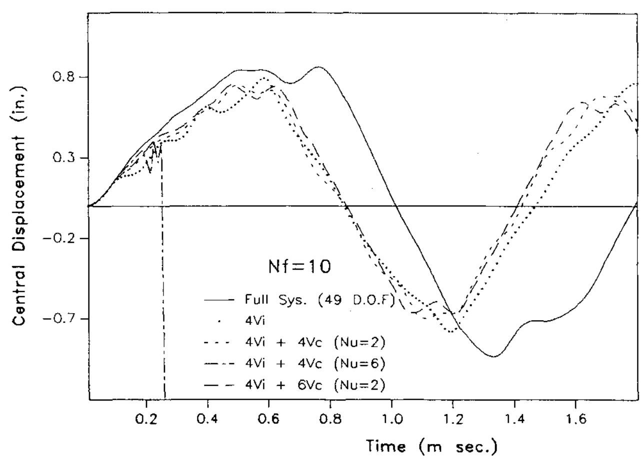

Fig. 2 Nonlinear dynamic response, effect of updating basis set, case 1.

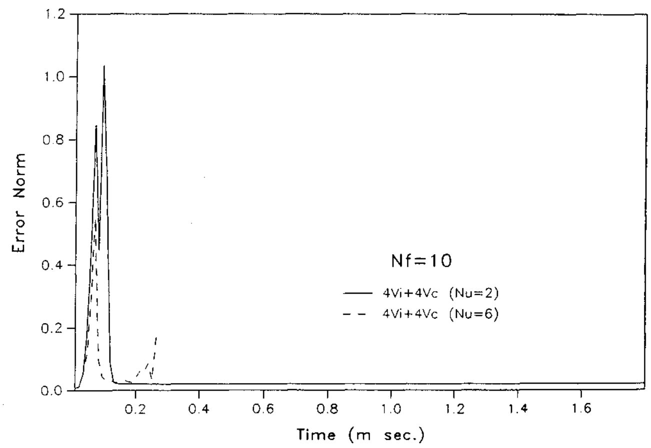

Fig. 3 Error norm with basis set updating, case 1.

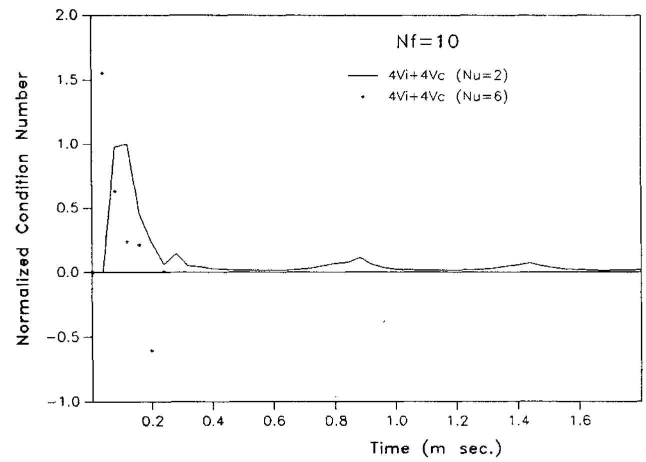

Fig. 4 Normalized condition number with basis updating, case 1.

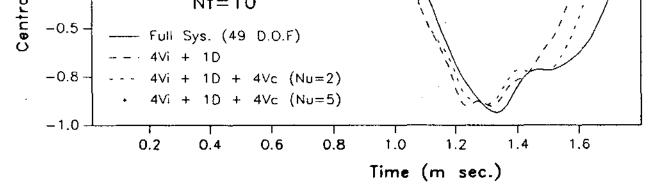

Fig. 5 Nonlinear dynamic response, effect of combining derivatives and updating, case 1.

basis set does not significantly improve the results. Also, when the frequency of updating the basis set approaches the frequency of stiffness matrix updating, the response becomes unstable. The error norm vs time response for the second and third set is shown in Fig. 3. When updating the basis vectors with a loose error criteria, the error norm decreases after the basis updatings. While updating with tight error criteria, the basis updating does not have the effect of causing the error norm to decrease. This suggests that an updating scheme which can cause a decreasing error norm is preferred. The normalized condition number vs time is given in Fig. 4. Updating has the effect of decreasing the condition number during response. While updating with a tight error criteria, the condition number may become zero or have a negative value. The zero or negative condition number causes the solution to diverge. This suggests that checking the condition number, before deciding to update, may be a useful method to avoid unstable solutions.

The response curves presented in Fig. 5 are similar to those in Fig. 2 except that one derivative vector is added to the basis. The addition of one derivative to the basis set results in the basis updating being much more beneficial. The error norm vs time response for the third and fourth set is shown in Fig. 6. Updating the basis vectors with a loose error criteria decreases the error norm. Updating the basis with a tight error criteria does not have the same effect. The normalized condition number vs time is given in Fig. 7. Here, updating the basis vectors with a tight error criteria, again produces a negative condition number and leads to instability. In fact, one derivative with four current vectors added to the initial basis set, gives similar results to those obtained by adding six derivatives to the initial basis set. Adding current Ritz vectors to the basis set while keeping the number of derivative basis vectors to a minimum seems to be a good approach, provided that a loose error criterion is used in the solution.

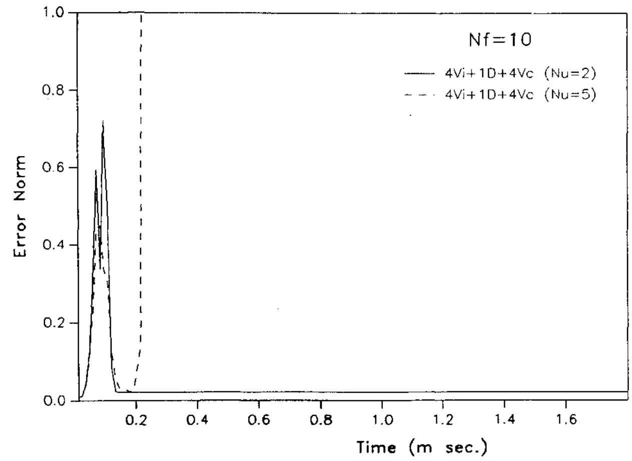

Fig. 6 Error norm with derivatives and updating, case 1.

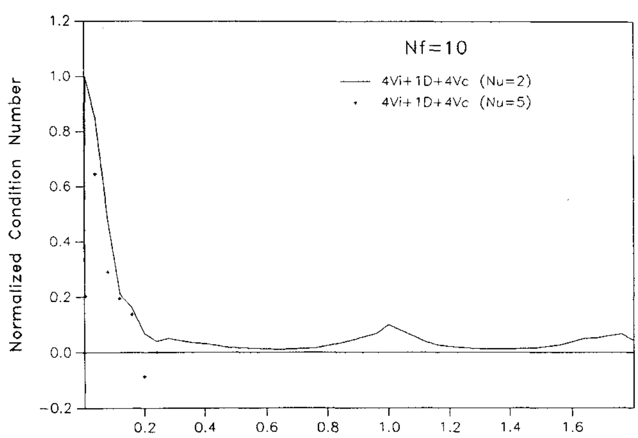

Fig. 7 Normalized condition number, derivatives and basis updating, case 1.

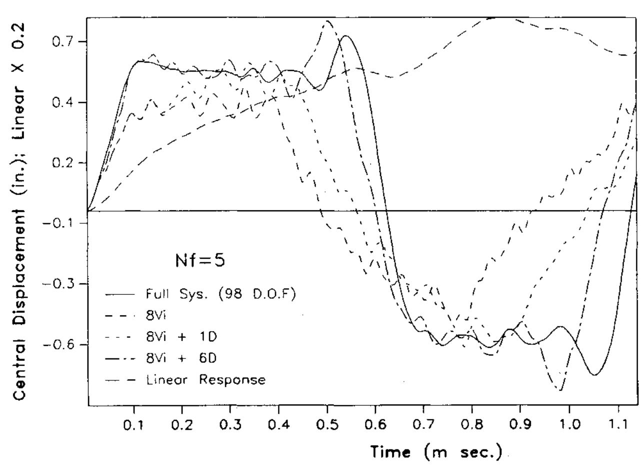

Fig. 8 Nonlinear dynamic response, effect of adding derivatives, case 2.

Case 2: Load Duration Near Eighth Natural Period

The finite element model for this case consists of ten elements and has 98 DOF. The specific parameters used for the response in this case are as follows: centrally applied load, 11,520 lbs; and time duration of load, 100 μs, which is about 116% of the eighth natural period of the undeformed beam and time integration step = 2 μs. In obtaining nonlinear response, the stiffness matrix is updated at every fifth time increment.

Nonlinear response for this particular loading are given in the following figures for this load case. In Fig. 8, results have been obtained for basis vectors with a different number of derivatives. The advantage of using more derivatives is again evident. In Fig. 9, results have been obtained for basis vectors with different current Ritz vectors. Again, the response curve cannot be improved by simply adding more current Ritz vectors to the basis set. Also, tightening the error norm again can have an adverse effect. The response curves presented in Fig. 10 are similar to those in Fig. 5 except for the shorter load duration. In Fig. 10, comparison of the nonlinear response for the basis set including initial Ritz vectors and one

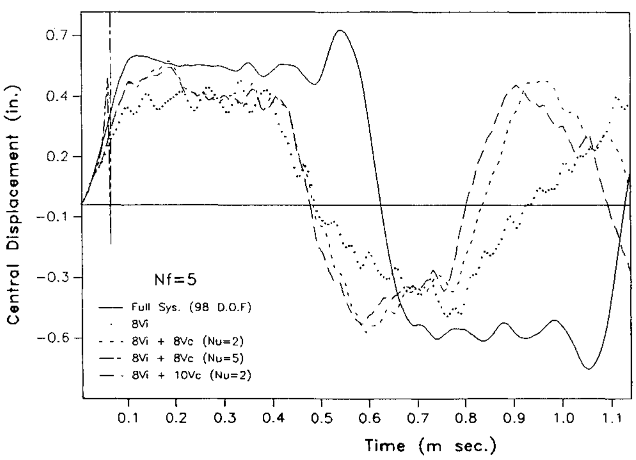

Fig. 9 Nonlinear dynamic response, effect of updating basis set, case 2.

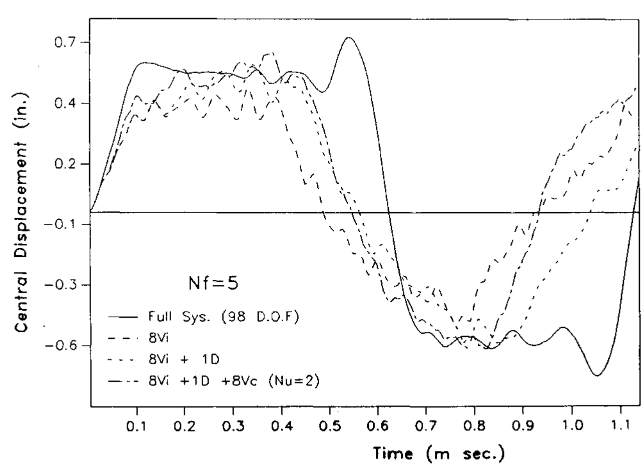

Fig. 10 Nonlinear dynamic response, effect of combining derivatives and updating, case 2.

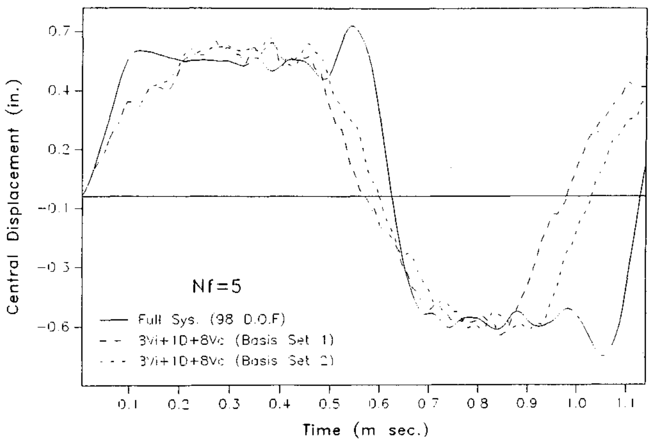

Fig. 11 Nonlinear dynamic response, different residual vibration era updating methods, case 2.

derivative vector, and for the basis set including initial Ritz vectors, one derivative vector and current Ritz vectors is shown. Comparing results for the loading duration near the fundamental and near the fourth period, it is evident that adding one derivative vector to current vectors is not as beneficial in the case of the higher frequency content. In this case, the time duration of loading is near the eighth period, more modes are expected to be involved in the response than when the loading time duration is near the lower period. Therefore more basis vectors, including more derivatives to account for the change of higher modes, are needed to accurately predict the response.

The effect of updating the basis vectors during the residual vibration era by using different approaches is shown in Fig. 11. Both basis sets use eight initial Ritz vectors, one derivative, and eight updated Ritz vectors. Both basis sets are updated once and updated at the same time step. The updating is in the residual vibration era. The first set uses Eq. (21) to generate the first updated Ritz vector, while the second set uses Eq. (26) to generate the first updated Ritz vector. Comparing the results, the response curve obtained using the second basis set provides a slightly more accurate solution. The first Ritz vector represents static displacement response due to the applied load. During the residual vibration era, there are no external loading forces applied to the system. The inertial forces would seem to be a better representation of the spatial loading distribution.

Conclusions

The results presented in this work indicate that the reduced basis approach using Ritz vectors is an effective method for determining the nonlinear response of impulsively loaded structural systems. Ritz vectors and their derivatives are chosen as basis vectors. The advantage of choosing this vector set is that they are effectively generated and can account for spatial load distribution.

In this work, initial Ritz vectors, derivative of Ritz vectors and updated Ritz vectors are used in combination to obtain solutions. The later two are chosen to account for the nonlinearities of the dynamic system. Results indicate that the response predicted using basis vectors with a suitable number of derivatives closely follows results obtained for the full system. Updating the basis vectors can also provide accurate solutions provided that derivative information is utilized to a limited extent. Efficiency is enhanced by updating the stiffness matrix at some predetermined reassembly frequency. Based on the numerical tests, updating the current basis vectors with a loose error criteria is a more desirable scheme than with a tight error criteria. It may be possible to use the condition number to avoid any unstable response behavior. While computational efficiency can not be demonstrated with models of the size studied here, efficiency should be significant for very large mathematical models, i.e., where the number of degrees of freedom is significantly greater than the number of basis vectors needed in the solution.

References

[1]Wilson, E. L., Yuan, M. W., and Dickens, J. M., "Dynamic Analysis by Direct Superposition of Ritz Vectors," *Earthquake Engineering and Structural Dynamics*, Vol. 10, 1982, pp. 813–821.

[2]Noor, A. K., and Peters, J. M., "Reduced Basis Technique for Nonlinear Analysis of Structures," *AIAA Journal*, Vol. 18, No. 4, 1980, pp. 455–462.

[3]McNamara, J. F., and Marcal, P. V., "Incremental Stiffness Method for Finite Element Analysis of the Nonlinear Dynamic Problems," *Proceedings of the International Symposium on Numerical and Computational Methods in Structural Mechanics*, Urbana, IL, 1971, pp. 107–129.

[4]Mondkar, D. P., and Powell, G. H., "Evaluation of Solution Schemes for Nonlinear Structures," *Computers and Structures*, Vol. 9, 1978, pp. 223–236.

[5]Horri, K., and Kawahara, M., "A Numerical Analysis of the Dynamic Response of Structures," *Proceedings of the 19th Japan National Congress for Applied Mechanics*, 1969, pp. 17–22.

[6]Nickell, R. E., "Nonlinear Dynamics by Mode Superposition," *Computer Methods in Applied Mechanics and Engineering*, 7, 1976, pp. 107–129.

[7]Morris, N. F., "The Use of Modal Superposition in Nonlinear Problems," *Computers and Structures*, 1077, pp. 65–72.

[8]Noor, A. K., "Recent Advances in Reduction Methods for Nonlinear Problems," *Computers and Structures*, Vol. 13, 1981, pp. 31–44.

[9]Idelsohn S. R., and Cardona, A., "A Local-Dependent Basis for Reduced Nonlinear Structural Dynamics," *Computers and Structures*, Vol. 20, No. 1–3, 1985, pp. 203–210.

[10]Chang, C. L., and Engblom, J. J., "Nonlinear Dynamic Response of Impulsively Loaded Structures: A Reduced Basis Approach," *AIAA Journal*, Vol. 29, No. 4, 1991, pp. 613–618.

[11]Bathe, K. K., *Finite Element Procedures in Engineering Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1982.